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SPREAD OF PLASTICITY FROM A STACK OF CRACKS UNDER MODE I CONDITIONS†

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Abstract—The paper studies the extent of plastic relaxation around the tips of an infinite sequence of slitlike cracks contained in a large elastic solid. The cracks have a constant distance of vertical separation, and the solid is deforming under tensile loading (mode I). The plastic region around each of the crack tips coplanar with the crack itself is represented by a suitable distribution of edge dislocations, which is determined from equilibrium considerations. The latter lead to a singular integral equation which is solved numerically. The solution procedure is uniformly valid for any crack spacing. Furthermore, an alternate perturbation technique is used for widely spaced cracks. Solutions are obtained as a function of the crack spacing and the applied tensile load, and the results discussed from the point of fracture initiation at stresss concentrations.

1. INTRODUCTION

The presence of multiple cracks in a large elastic body can significantly alter its gross elastic response and fracture characteristics. The degree of such a change depends on the crack spacing, their relative orientation and the far-field state of stress. An exact analytical investigation of the physical situation would require an allowance for the randomness of the orientation and of the distribution and scale of the cracks. Such an investigation is, however, an almost impossible task because the mapping technique normally used in solving the simplest crack problems presents formidable mathematical difficulties. Nonetheless, a fairly accurate picture can be obtained by studying certain idealized models, viz. those of homogeneous and isotropic elastic solids containing a regular distribution of cracks.

Many authors, among them Louat[1], Smith[2], Bilby, Cottrell and Swinden[3], Dugdale[4], Rice[5], Kostrov and Nikitin[6], Field[7], Koiter[8], Paris and Sih[9] and Isida[19], considered various simplified models of single or multiple cracks in an infinite elastic solid with a view to studying the change in the elastic properties of the solid and/or plastic relaxation around the cracks, the latter being useful in investigating the fracture characteristics of the solid. (For a more detailed historical development of this subject, see Karihaloo[10]).

A more difficult problem of an infinite solid with a doubly periodic array of unrelaxed slitlike cracks was studied in detail under all the three modes of loading (anti-plane shear, mode III; in-plane shear, mode II; tensile loading, mode I) by Delameter, Herrmann and Barnett[11] and Delameter and Herrmann[12]. Characteristically, these authors employed in their study the equivalence of slitlike cracks (displacement discontinuities) and straight dislocations to overcome the formidable mathematical difficulties associated with the mapping technique. The technique of modelling cracks by dislocations was used by Karihaloo[10] in studying the stress relaxation round non-coplanar inhomogeneities (cracks) in an infinite solid under mode II loading.

The present work is an extension of this technique to solids under the more frequently met tensile loading conditions. Thus, Section 2 models the infinite sequence of cracks $|x| \leq c$, $y = \pm nh$ ($n = 0, 1, 2$, etc.) in an infinite elastic solid by suitable distributions of edge dislocations. The body is subjected to an external tensile stress $\sigma_{yy} = \sigma$ at infinity. The singular integral equation governing the equilibrium of the distributed dislocations, unlike the mode III case, does not seem to have a closed form solution. Therefore, an approximate numerical procedure, like the one used in[10], is employed after expanding the non-singular part of the kernel in a series of Chebyshev polynomials (Section 3). The models of an infinite sequence of non-coplanar unrelaxed cracks (Benthem and Koiter[13]) and an isolated relaxed crack[3] under tensile loading form particular cases of the present study. In Section 4 an alternate perturbation technique, similar to the one used by Smith[2], is used to obtain the solution of the problem for the special case of widely

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spaced cracks. Finally, in Section 5 the opening mode I results are used to discuss briefly the initiation of fracture at stress concentrations accentuated by closely spaced cracks.

2. THE TENSILE LOADING MODEL (MODE I)

To study the plastic relaxation at the tips of an infinite sequence of slitlike cracks $|x| \leq c$, $y = \pm nh$ contained in an infinite elastic solid subjected to an external tensile stress $\sigma_{yy} = \sigma$, the displacement discontinuity across each crack in the y-direction is represented by straight edge dislocations parallel to x-axis and lying in the planes $y = \pm nh$. Under an external stress σ_{yy} the edge dislocations are said to be in "climb", as opposed to "glide" under stress σ_{xy}. The plastic zones around the tips of the cracks coplanar with cracks themselves are represented by similar dislocations Fig. 1. However, the movement of the latter is hindered by frictional stresses taken to be equal to the yield stress σ_y of the material. Thus, if the plastic zones spread out to a distance a, the dislocations modelling these zones $c < |x| \leq a$, $y = \pm nh$ are subject to a net stress $\sigma_{yy} = \sigma - \sigma_y$, in addition to the interaction stress from all other dislocations, while those modelling the freely slipping cracks $|x| < c$, $y = \pm nh$ are subject only to the stress $\sigma_{yy} = \sigma$, besides the various interaction stresses from other dislocation arrays.

To evaluate these interaction stresses the edge dislocations modelling the cracks are treated as vertical arrays. The stress σ_{yy} due to such an array of positive edge dislocations situated along the plane $x = x'$ (see, for example, Weertman and Weertman[14]) is

$$\sigma_{yy} = -\frac{\mu b}{2\pi(1-\nu)} \sum_{n=-\infty}^{\infty} \frac{nh[(x-x')^2 - n^2h^2]}{[(x-x')^2 + n^2h^2]^2}, \tag{1}$$

at a point x along one of the planes $y = \pm nh$. Here μ is the shear modulus, b the Burgers vector of each edge dislocation and ν Poisson's ratio. The expression (1) for σ_{yy} can be simplified by evaluating the infinite sum over n in terms of hyperbolic functions

$$\sigma_{yy} = \frac{\mu b}{2\pi(1-\nu)} \frac{1}{(x-x')} \left[\frac{\pi(x-x')/h \sinh 2\pi(x-x')/h - \pi^2(x-x')^2/h^2}{\sinh^2 \pi(x-x')/h} \right]. \tag{2}$$

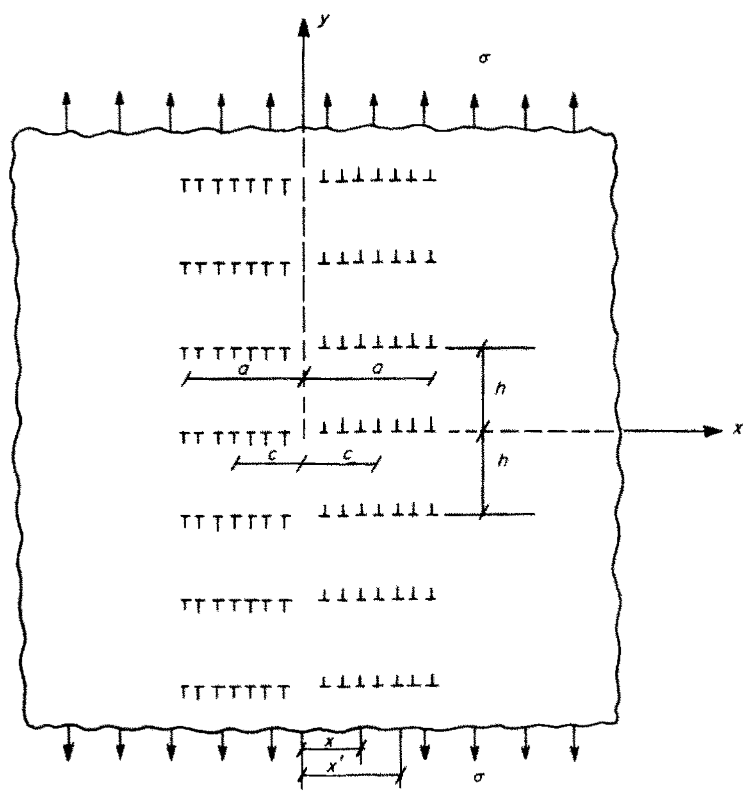

Fig. 1. Equivalent dislocation distributions simulating the cracks and coplanar plastic zones.

Because of the assumed regular pattern of cracks the distribution of dislocations representing each of the cracks will be the same. However, it is assumed that the dislocations in each crack are smeared out such that there are $f(x)\,dx$ dislocations each of Burgers vector b in any interval dx. Hence, the integral equation, which expresses the requirement that the resultant stress on any dislocation in the distribution vanish when the system is in equilibrium (Bilby and Eshelby[15]), can be obtained by summing the effects due to all other dislocations, and is

$$\int_{-a}^{a} \frac{\mu b}{2\pi(1-\nu)} \frac{1}{(x-x')} \left[\frac{\pi(x-x')/h \sinh 2\pi(x-x')/h - \pi^2(x-x')^2/h^2}{\sinh^2 \pi(x-x')/h}\right] f(x')\,dx' + F(x) = 0 \quad (3)$$

for $|x| \leq a$, where $F(x)$ is the net applied stress at x. As mentioned above, this singular integral equation, as opposed to the corresponding equation for anti-plane strain mode III, does not seem to have a closed form solution. A perturbation solution will be presented in Section 4 for the less interesting case of widely spaced cracks. For the more important case of closely spaced cracks we employ the efficient approximate method used earlier in[10–12]. This method is uniformly valid for all values of crack spacing. Before presenting some salient features of this method, we introduce non-dimensional variables.

Let $x_1 = x/a, x_1' = x'/a, h_1 = h/\pi c, f(x_1') = \mu b f(x')/2\pi(1-\nu)\sigma_y$ and denote $\alpha = c/a \leq 1$ (equality holding for unrelaxed cracks). Then eqn (3) takes the form (subscript 1 has been omitted in sequel)

$$\int_{-1}^{1} \frac{1}{(x-x')} \left[\frac{(x-x')/\alpha h \sinh 2(x-x')/\alpha h - (x-x')^2/\alpha^2 h^2}{\sinh^2 (x-x')/\alpha h}\right] f(x')\,dx' + P(x) = 0, \quad (4)$$

where $P(x) = \sigma/\sigma_y$ in the freely slipping crack interval $0 \leq |x| \leq \alpha$ and $P(x) = \sigma/\sigma_y - 1$ in the plastic zones ahead of the crack tips $\alpha < |x| \leq 1$.

For future use, (4) is rewritten as

$$\int_{-1}^{1} \left\{\frac{1}{x-x'} + K(x',x)\right\} f(x')\,dx' + P(x) = 0 \quad (5)$$

where the non-singular part $K(x', x)$ of the kernel is given by the expression

$$K(x',x) = \frac{1}{(x-x')} \left[\frac{(x-x')/\alpha h \sinh 2(x-x')/\alpha h - (x-x')^2/\alpha^2 h^2}{\sinh^2 (x-x')/\alpha h} - 1\right]. \quad (6)$$

It is easily shown that, as $x \to x'$, $K(x', x) \to 0$. We are required to solve (5) in order to determine $f(x)$ and α (the relation between c and a) for various values of the physical parameters σ_y and σ as a function of the crack spacing h.

3. METHOD OF SOLUTION

The approximate method used to solve the singular integral eqn (5) is similar to that used in[10]. We, therefore, present only the most necessary expressions here. The crux of this method lies in expanding the non-singular part of the kernel (6) in a series of Chebyshev polynomials in the variable x, the coefficients of the series being functions of x'. Thus for given values of h and σ/σ_y (of course, α and σ/σ_y are related, see below), $K(x', x)$ may be expanded as

$$K(x',x) = \sum_{n=0}^{\infty} A_n(x') T_n(x), \quad (7)$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind. From the orthogonality properties of T_n, it follows that

$$A_0(x') = \frac{1}{\pi} \int_{-1}^{1} K(x',x)/\sqrt{(1-x^2)}\,dx$$

$$A_n(x') = \frac{2}{\pi} \int_{-1}^{1} K(x',x) T_n(x)/\sqrt{(1-x^2)}\,dx; \quad n = 1, 2, 3, \ldots. \quad (8)$$

Substituting (7) into (5) and rearranging the terms, we have

$$\int_{-1}^{1} \frac{f(x')}{x - x'}\, dx' = \sum_{n=0}^{\infty} a_n T_n(x) - P(x), \tag{9}$$

where

$$a_n = -\int_{-1}^{1} f(x') A_n(x')\, dx'. \tag{10}$$

The condition for the vanishing of $f(x)$ at the tips of the plastic zones ($x = \pm 1$) ahead of the cracks is [16]

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} \frac{T_n(x)}{\sqrt{(1-x^2)}}\, dx - \int_{-1}^{1} \frac{P(x)}{\sqrt{(1-x^2)}}\, dx = 0. \tag{11}$$

The condition (11) of the existence of a solution to (10) specifies the distance to which the yield spreads under a given stress σ. In other words it gives a relation between α and σ/σ_y. In fact, as a consequence of certain orthogonality properties of T_n [10], it is easily shown that (11) reduces to

$$\cos^{-1} \alpha = \frac{\pi}{2}\left(\frac{\sigma}{\sigma_y} - a_0\right). \tag{12}$$

A comparison with the corresponding expression for an isolated relaxed crack [3, 4] easily shows that the presence of an infinite sequence of relaxed cracks manifests itself through the additional term $(\pi/2)a_0$ in the right hand side of (12).

When the existence condition (11) is fulfilled, solution of eqn (9) gives the distribution function

$$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x) + \eta(x), \tag{13}$$

where $\eta(x)$—identical to the solution for an isolated relaxed crack—is given by

$$\eta(x) = \frac{1}{\pi^2}\left[\cosh^{-1}\left|\frac{1-\alpha x}{\alpha - x}\right| - \cosh^{-1}\left|\frac{1+\alpha x}{\alpha + x}\right|\right]. \tag{14}$$

The functions $\psi_n(x)$ are easily shown to be

$$\psi_0(x) = 0,$$

$$\psi_{n+1}(x) = \frac{\sqrt{(1-x^2)}}{\pi} U_n(x); \quad n = 0, 1, 2, \ldots\ldots, \tag{15}$$

where $U_n(x)$ is the nth Chebyshev polynomial of the second kind. It can be shown that $f(x)$ is an odd function. Consequently, given that $\psi_0(x) = 0$ and $\eta(x)$ is an odd function, (13) may be rewritten as

$$f(x) = \sum_{n=1}^{\infty} b_n \psi_{2n}(x) + \eta(x),$$

where

$$\psi_{2n}(x) = \frac{\sqrt{(1-x^2)}}{\pi} U_{2n-1}(x) \qquad \text{is an odd function,}$$

and

$$b_n = -\int_{-1}^{1} f(x') A_{2n}(x')\, dx'.$$

The unknown coefficients b_n are determined by substituting $f(x)$ into the expression for b_n. This results in an infinite system of linear algebraic equations

$$\sum_{j=1}^{\infty} C_{ij} b_j = D_i, \qquad i = 1, 2, 3, \ldots\ldots, \tag{16}$$

where

$$D_i = -\int_{-1}^{1} \eta(x') A_{2i}(x')\,\mathrm{d}x'$$

and

$$C_{ij} = \delta_{ij} + \int_{-1}^{1} \psi_{2j}(x') A_{2i}(x')\,\mathrm{d}x', \tag{17}$$

δ_{ij} being the Kronecker delta. Having determined the coefficients b_n and, hence, the function $f(x)$, the original coefficient a_0 entering (12) is evaluated from

$$a_0 = -\int_{-1}^{1} f(x) A_0(x)\,\mathrm{d}x.$$

From the point of view of studying the fracture characteristics of a solid containing an infinite sequence of cracks using the crack opening displacement criterion it is necessary to know the relative displacement $\Delta(x)$ of the faces of the crack given by

$$\Delta(x) = b[N(1) - N(x)], \tag{18}$$

where b is the Burgers vector of an individual dislocation and $N(x)$ is the number of dislocations between 0 and x obtained by integrating $f(x)$ over this interval. If this is done, we obtain

$$\frac{\Delta(x)}{b} = \sum_{n=1}^{\infty} b_n\left[\int_0^1 \psi_{2n}(x)\,\mathrm{d}x - \int_0^x \psi_{2n}(x)\,\mathrm{d}x\right]$$
$$+\frac{1}{\pi^2}(x+\alpha)\cosh^{-1}\left|\frac{1+\alpha x}{\alpha + x}\right| - \frac{1}{\pi^2}(x-\alpha)\cosh^{-1}\left|\frac{1-\alpha x}{\alpha - x}\right|. \tag{19}$$

Finally, the crack tip ($|x| = \alpha$) opening displacement in dimensionless quantities is given by

$$\Delta^*(\alpha) = \frac{\pi^2}{2\alpha}\sum_{n=1}^{\infty} b_n \int_{\alpha}^{1} \psi_{2n}(x)\,\mathrm{d}x + \ln\left(\frac{1}{\alpha}\right), \tag{20}$$

where $\Delta^*(x) = \Delta(x)\pi\mu/4c(1-\nu)\sigma_y$, and $\alpha = c/a$ is specified by (12). Again, the corresponding expression for an isolated crack is easily recovered.

The coefficients b_n were determined for given values of α and h. The linear system of algebraic equations in b_n (16) was truncated at $i = j = 12$. All integrals were evaluated by Simpson's rule after making a change of variables in some integrals to render the integrands non-singular. The value of σ/σ_y corresponding to the assumed value of α was evaluated from (12).

The distance to which plasticity spreads from the tips of each of the cracks as a function of the applied stress σ and the crack spacing $k = h/\pi c$ is shown in Fig. 2 and compared with that for an isolated crack. The results for widely spaced cracks are in good agreement with those obtained by a perturbation technique (designated by dots).

4. PERTURBATION SOLUTION FOR WIDELY SPACED CRACKS

We present here a simple perturbation solution of $f(x')$ for widely spaced cracks. This method is similar to that used in[2].

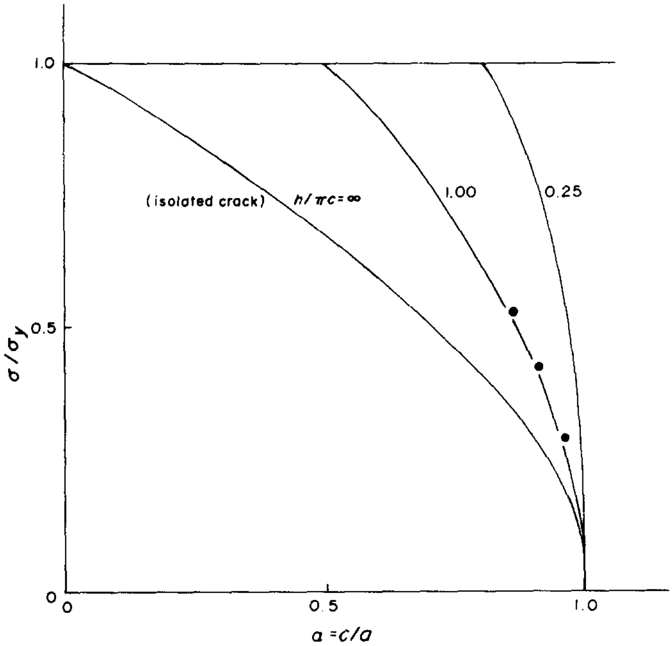

Fig. 2. The extent of spread of plasticity a under stress $\sigma_{yy} = \sigma$ for different values of crack spacing h. c, half the crack length, σ_y, yield stress.

Let the vertical separation between the cracks be such that $\epsilon = a/h \ll 1$. Then $f(x')$ in (3) may be expressed in the form of a power series in the small parameter ϵ

$$f(x') = \sum_{t=0}^{\infty} \epsilon^t f_t(x'), \tag{21}$$

where $f_t(x')$ are functions of x', but independent of h. It is easily shown that for $\epsilon \ll 1$,

$$\left[\frac{\pi(x-x')/h \sinh 2\pi(x-x')/h - \pi^2(x-x')^2/h^2}{\sinh^2 \pi(x-x')/h}\right] = 1 + \epsilon^2[\pi(x-x')/a]^2 + 0(\epsilon^4). \tag{22}$$

Substituting (21) and (22) into (3) and equating the coefficients of like powers in ϵ, we have

$$\begin{aligned}
&\int_{-a}^{a} \frac{f_0(x')}{x-x'}\,dx' + \frac{2\pi(1-\nu)}{\mu b} F(x) = 0,\\
&\int_{-a}^{a} \frac{f_1(x')}{x-x'}\,dx' = 0,\\
&\int_{-a}^{a} \frac{f_2(x')}{x-x'}\,dx' + \frac{\pi^2}{a^2}\int_{-a}^{a} (x-x') f_0(x')\,dx' = 0.
\end{aligned} \tag{23}$$

These three singular integral equations are easily solved using the inversion formula [16]

$$\begin{aligned}
f_0(x) &= -\frac{2(1-\nu)}{\pi\mu b}\frac{1}{\sqrt{(a^2-x^2)}}\int_{-a}^{a}\frac{\sqrt{(a^2-x'^2)}F(x')\,dx'}{(x'-x)},\\
f_1(x) &= 0,\\
f_2(x) &= -\frac{2\pi(1-\nu)}{\mu b a^2}\frac{x}{\sqrt{(a^2-x^2)}}\int_{-a}^{a}\sqrt{(a^2-x'^2)}F(x')\,dx',
\end{aligned} \tag{24}$$

whence it follows that

$$f(x) = f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + 0(\epsilon^4)$$
$$= -\frac{2(1-\nu)}{\pi\mu b}\left[\frac{1}{\sqrt{(a^2-x^2)}}\int_{-a}^{a}\frac{\sqrt{(a^2-x'^2)}F(x')\,dx'}{(x'-x)} + \frac{\pi^2}{a^2}\left(\frac{a}{h}\right)^2\frac{x}{\sqrt{(a^2-x^2)}}\int_{-a}^{a}\sqrt{(a^2-x'^2)}F(x')\,dx'\right] + 0\left(\frac{a}{h}\right)^4, \quad (25)$$

where $F(x) = \sigma$; $|x| \leq c$
$= \sigma_y - \sigma$; $c < |x| \leq a$.

The distance to which plasticity spreads from the tips of the crack $|x| = c$ is determined from the condition that $f(x)$ vanish at the tips of plastic zones $|x| = a$. When this condition and the symmetry consideration are taken into account, (25) gives

$$\int_0^a \frac{F(x)\,dx}{\sqrt{(a^2-x^2)}} - \frac{\pi^2}{a^2}\left(\frac{a}{h}\right)^2\int_0^a\sqrt{(a^2-x^2)}F(x)\,dx = 0, \quad (26)$$

which, after simplification, takes the form

$$\cos^{-1}(\alpha) = \frac{\pi}{2}\frac{\sigma}{\sigma_y} - \frac{(1-\alpha^2)^{1/2}}{2\alpha k^2}, \quad (27)$$

where, as before, $\alpha = c/a$, $k = h/\pi c$. When the condition (27) is satisfied, eqn (25) gives the distribution function

$$f(x) = -\frac{2(1-\nu)}{\pi\mu b}\sqrt{(a^2-x^2)}\int_{-a}^{a}\frac{F(x')}{\sqrt{(a^2-x'^2)}(x'-x)}\,dx' + 0\left(\frac{a}{h}\right)^4$$
$$= \frac{2\sigma_y(1-\nu)}{\pi\mu b}\sinh^{-1}\left|\frac{2x(a-x)c(a-c)^{1/2}}{a(c-x)}\right| + 0\left(\frac{a}{h}\right)^4. \quad (28)$$

As $h \to \infty$, the model becomes that of an isolated relaxed crack in an infinite solid, and eqns (27) and (28) reduce to the corresponding expressions of [3, 4].

If, on the other hand, σ_y is allowed to become infinite the model represents an infinite sequence of parallel unrelaxed cracks (Benthem and Koiter, [13]). In this case expression (27) becomes meaningless, since $a = c$, but the distribution function (25) becomes

$$f(x) = \frac{2\sigma(1-\nu)}{\mu b}\frac{x}{\sqrt{(c^2-x^2)}}\left(1 - \frac{\pi^2}{2}\frac{c^2}{h^2}\right) + 0\left(\frac{c}{h}\right)^4, \quad (29)$$

which is in good agreement with the result of Benthem and Koiter. As before, the crack tip opening displacement

$$\Delta(c) = b \cdot N(c),$$

where $N(c)$ is the number of edge dislocations in each half of one of the planes $y = \pm nh$ which is obtained by integrating (29) with respect to x between the limits 0 and c, whereupon

$$\Delta(c) = \frac{2(1-\nu)\sigma c}{\mu}\left(1 - \frac{\pi^2}{2}\frac{c^2}{h^2}\right) + 0\left(\frac{c}{h}\right)^4. \quad (30)$$

5. DISCUSSION

From Fig. 2 it is seen that the tensile stress σ required to spread the plasticity a given distance from the crack tip increases with decreasing distance between the cracks. In other words, the effective stress-intensity factor decreases with a reduction of the crack spacing (h). This follows from the fact that the number of dislocations in each half of one of the planes (eqn 19) $y = \pm nh$, being a measure of the crack opening displacement, decreases as the cracks come closer. This is in agreement with the results for unrelaxed cracks reported in[11, 12]. However, it is interesting to note that this phenomenon is exactly the same as observed in anti-plane shear mode III but just the opposite of that for the in-plane shear mode II.

To get some idea of the fracture characteristics of an infinite solid containing an infinite sequence of cracks under mode I conditions, it is sufficient to consider the present resuts in the light of the crack opening displacement criterion. Accordingly, if the fracture is supposed to initiate when the relative opening of the crack tips $\Delta(c)$ exceeds some critical value (a material property), it is evident from Fig. 3 that the tendency towards fracture decreases as the cracks come closer. This follows from the fact that, for a given applied stress σ, the distance a decreases as the distance between the cracks decreases. In other words, the presence of a stack of cracks improves the fracture characteristics of the solid in comparison with that of an isolated crack. This conclusion is just the opposite of that arrived at in the in-plane shear mode II.

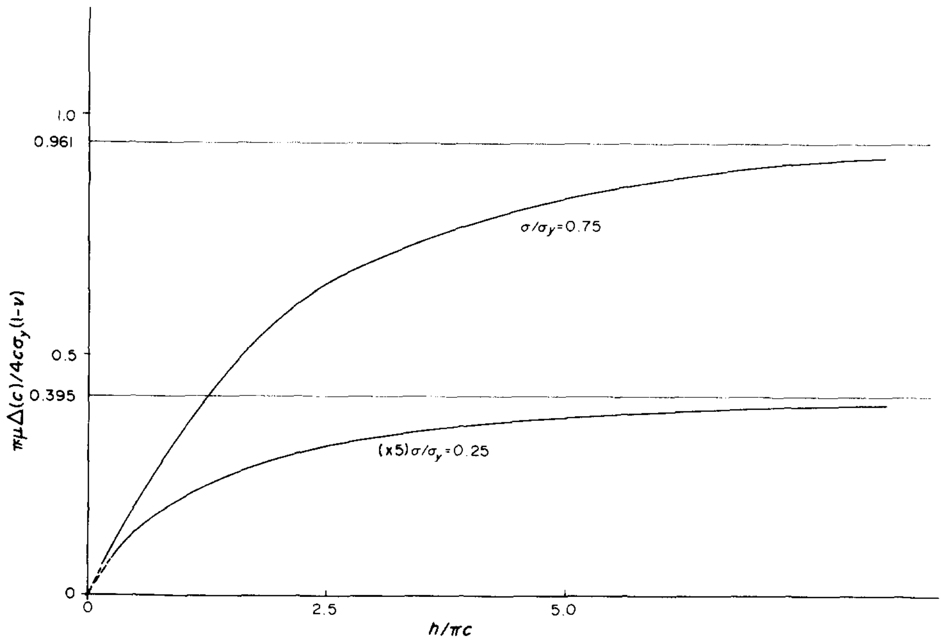

Fig. 3. Crack tip opening displacement $\Delta(c)$ as a function of crack spacing h for various values of external stress $\sigma_{yy} = \sigma$. The asymptotes correspond to an isolated crack ($h = \infty$). Note that the ordinates have been magnified fivefold for $\sigma/\sigma_y = 0.25$.

Finally, it should be mentioned that the present model in which the plastic zones are collinear with the cracks leaves much to be desired in mode I situation. A more realistic configuration would be one in which the plastic bands are allowed to spread off the planes of the cracks. Such a model was considered numerically by Bilby and Swinden[17] for an isolated crack with the plastic bands inclined at an angle of 45° to the plane of the crack. Subsequently, it was generalized by Karihaloo[18] to take into account in-plane component of stresses.

In connexion with a stack of cracks with plastic bands inclined to the crack planes it would be necessary to account for the interaction of plastic zones.

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